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Half-Moment Closure for Radiative Transfer Equations[1]

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In this paper a moment method for radiative transfer equations is considered which has been developed and investigated using different approaches. Problems appearing for this moment system for boundary value problems using Maxwell-type boundary conditions are described. A new method based on the consideration of positive and negative half fluxes is developed and shown to overcome the above problems. Moreover, a numerical scheme and numerical results for the new moment system are presented.

Key Words: moment methods; entropy minimization; half fluxes; Maxwell boundary conditions.

1. INTRODUCTION

Radiative transfer equations are used in many applications, for example to describe the cooling of molten glass or heat transfer in gas turbines. For most of these applications extensive computation time is needed to solve the equations. Therefore, approximate methods have been developed. Approximate equations can be derived from the full transport equations by asymptotic analysis or simply by taking suitable moments and closure relations. Examples are diffusion or Rosseland equations, the P_N equations, and moment equations closed by the entropy minimization principle [2, 6, 8–10, 12]. The latter have turned out to describe certain physical situations, i.e., solutions of the full transport equation, much better than diffusion-type equations (see [6, 12]). For restrictions see [4, 8]. In the following we give an example where these equations give nonphysical results and suggest a new set of moment equations without these difficulties.

We consider the equations for radiative heat transfer in a simplified setting. It should be stressed that the present simplification is not essential for the method. Other physical situations can be considered and are discussed at the end of the paper.

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More exactly we consider equations for the radiative intensity $I(x, \mu, \nu, t)$, where space, angular direction, and frequencies are denoted $x \in [0, 1]$, $\mu \in [-1, 1]$, $\nu \in [0, \infty)$,

$$\epsilon \partial_t I + \mu \partial_x I = \frac{1}{\varepsilon}(B(I) - I), \tag{1}$$

where

$$B(I) = \frac{15}{\pi^4} \frac{\nu^3}{\exp(c\nu) - 1}, \quad \text{with } c = \langle I \rangle^{-1/4},$$

with the definition

$$\langle \cdot \rangle = \frac{1}{2} \int_0^{+\infty} \int_{-1}^{+1} \cdot \, d\mu \, d\nu.$$

We note that

$$\int_0^{+\infty} \frac{\nu^3}{\exp(c\nu) - 1} \, d\nu = \frac{\pi^4}{15 c^4}$$

and therefore

$$\langle B(I) \rangle = \langle I \rangle.$$

Boundary conditions are, for example, given by prescribing the ingoing fluxes at $x = 0$ and $x = 1$:

$$I_b|_{x=0, \mu>0}, \quad I_b|_{x=1, \mu<0}.$$

The simple choice of the boundary conditions is not important for the approach presented in the following. More-complicated wall models like reflecting boundary conditions can be treated as well.

Moment systems have been obtained for these equations by multiplying the equations, for example, with $m(\mu) = (1, \mu)$, i.e.,

$$\left\langle \left[\epsilon \partial_t I + \mu \partial_x I - \frac{1}{\varepsilon}(B(I) - I) \right] m(\mu) \right\rangle = 0, \tag{2}$$

and closing the equations with an entropy minimization principle, [2, 6, 9, 10]. The minimization principle yields the distribution function

$$\mathcal{B} = \frac{15}{\pi^4} \frac{\nu^3}{\exp(\nu \alpha (1 + \beta \mu)) - 1}, \tag{3}$$

with

$$\int_0^{\infty} \mathcal{B} \, d\nu = \frac{1}{(\alpha(1 + \beta\mu))^4},$$

and the parameters α and β are determined by fixing the first two moments

$$E = \langle 1I \rangle,$$
$$F = \langle \mu I \rangle.$$

This gives $\mathcal{B} = \mathcal{B}(E, F)$. Using this distribution function to close the moment equations gives the closure

$$\langle \mu^2 I \rangle \sim P = \langle \mu^2 \mathcal{B} \rangle = \chi E,$$

with the Eddington factor

$$\chi = \chi(f), \quad f = \frac{F}{E}, \quad f \in [-1, 1],$$

and

$$\chi = \frac{3 + 4f^2}{5 + 2\sqrt{4 - 3f^2}} \in \left[\frac{1}{3}, 1\right].$$

The moment system (2) reads finally

$$\begin{aligned} \epsilon \partial_t E + \partial_x F &= 0, \\ \epsilon \partial_t F + \partial_x \chi E &= -\frac{1}{\epsilon} F. \end{aligned} \tag{4}$$

We note that the above closure has to be considered carefully, Restrictions of this equilibrium closure are mentioned, for example, in [4, 8].

It can be shown that the absolute value of the relative flux f is always bounded by 1. The eigenvalues of the system are easily evaluated and found to have absolute values also bounded by 1. One finds two negative eigenvalues for f below some critical value $-f_C$, where f_C is equal to $\frac{2\sqrt{3}}{5}$. Moreover, one obtains one negative and one positive eigenvalue for $f \in [-f_C, f_C]$ and two positive eigenvalues for f above f_C (see, for example, [6]).

We use the Maxwell procedure, i.e., equalizing ingoing half range fluxes at the boundary, to find boundary conditions for our moment system. For $x = 0$ they are

$$\int_0^\infty \int_{\mu>0} \mu I_b m(\mu)\, d\mu\, d\nu = \int_0^\infty \int_{\mu>0} \mu \mathcal{B} m(\mu)\, d\mu\, d\nu, \tag{5}$$

with $m = \binom{1}{\mu}$. One proceeds analogously for $x = 1$ with $\mu < 0$. Depending on the number of ingoing characteristics for system (4) zero, one or two conditions are prescribed on each side.

We consider an example with equilibrium boundary conditions

$$I_b = \frac{15}{\pi^4} \frac{\nu^3}{\exp\left(\nu \delta^{-1/4}\right) - 1}, \tag{6}$$

with

$$\int_0^{+\infty} I_b \, d\nu = \delta.$$

For the following discussion we consider at $x = 0$ the above boundary data with $\delta = \delta_l = 0$. For $x = 1$ we choose $\delta = \delta_r = 1$. The numerical results show for the full transport solution that the relative flux $f = \frac{\langle \mu I \rangle}{\langle I \rangle}$ for this problem is always in the subcritical range $[-f_C, f_C]$ (see Section 3). In this case one boundary condition has to be imposed for the moment system on the left side of the computational domain. Using the first Marshak condition we get for $x = 0$ (where δ was equal to 0)

$$\int_0^\infty \int_{\mu>0} \mu \mathcal{B} \, d\mu \, d\nu = 0.$$

Looking at the exact form of the distribution function $\mathcal{B}$ (compare [6]), one observes that this can only be true for $f = -1$. In this case $\mathcal{B}$ is singular at $\mu = -1$. Obviously, this contradicts the assumption that the relative flux is in $[-f_C, f_C]$. The moment system with these conditions is obviously inconsistent with the solution of the full transport equation.

The difficulty discussed here can be partially avoided by using an approach similiar to kinetic schemes (see, e.g., [11]). The kinetic boundary conditions can be included in the numerical solution of a full space moment model in different ways. One can either restrict the kinetic approach to the boundary and use a upwind scheme based on Roe solvers, etc., or use a kinetic scheme in the whole computational domain (see [1, 7, 13] for such approaches). We will show that the new approach presented in the following includes the boundary condition in a natural way in the model and, moreover, is more accurate than the approach described above.

We note that considering the transport or the moment equations as ϵ tends to 0, one obtains the diffusion approximation. The Maxwell procedure gives in this case the so-called slip boundary conditions. These conditions yield well-known reasonable approximations of the full transport solution. They are used for comparison in the numerical examples below. More exactly one obtains the limit equations

$$\partial_t E - \partial_x^2 \frac{1}{3} E = 0. \tag{7}$$

The distribution function to be used in the Maxwell procedure (5) is a polynomial expansion, given by expanding the full transport solution or the function $\mathcal{B}$ (noting that f, F are of order ϵ):

$$\mathcal{B}_{expanded} = E - \epsilon \mu \partial_x E + \cdots. \tag{8}$$

We note that in this simple case the diffusion equation with slip boundary conditions is equivalent to the P_1 approximation with Maxwell boundary conditions.

2. A NEW MOMENT SYSTEM BASED ON HALF FLUXES

To be able to include correctly the kinetic boundary conditions in the moment model we use half fluxes instead of full space moments as the basic quantities of our moment system (see, for example, [3] and references therein for a similiar approach in gas dynamics). We use the moments $m = (1^+, 1^-, \mu^+, \mu^-)$, where

$$1^- = 1|_{\mu<0}, \quad 1^+ = 1|_{\mu>0},$$
$$\mu^+ = 1^+\mu, \quad \mu^- = 1^-\mu.$$

Using the entropy minimization principle, fixing the half space moments, the distribution function, instead of (3), is now given by

$$\mathcal{B} = \frac{15}{\pi^4} \frac{\nu^3}{\exp(\nu(\alpha_-(1^- + \beta_-\mu^-) + \alpha_+(1^+ + \beta_+\mu^+)) - 1)}.$$

Note that

$$\int_0^{+\infty} \mathcal{B}\, d\nu = \frac{1}{(\alpha_-(1^- + \beta_-\mu^-) + \alpha_+(1 + \beta_+\mu^+))^4},$$

where the parameters

$$\beta_- < 1, \quad \beta_+ > -1, \quad \alpha_- > 0, \quad \alpha_+ > 0$$

are given by fixing the half space moments

$$\begin{aligned} E_- &= \langle 1^- I \rangle, \\ E_+ &= \langle 1^+ I \rangle, \\ F_- &= \langle \mu^- I \rangle, \\ F_+ &= \langle \mu^+ I \rangle. \end{aligned}$$

We have $\mathcal{B} = \mathcal{B}(E_-, E_+, F_-, F_+)$ and

$$\begin{aligned} E &= \langle I \rangle = E_+ + E_-, \\ F &= \langle \mu I \rangle = F_+ + F_-. \end{aligned}$$

Using the above distribution function to close the moment equations gives the closure

$$\begin{aligned} \langle I(\mu^+)^2 \rangle &\sim P_+ = \langle \mathcal{B}(\mu^+)^2 \rangle = \chi_+ E_+, \\ \langle I(\mu^-)^2 \rangle &\sim P_- = \langle \mathcal{B}(\mu^-)^2 \rangle = \chi_- E_-, \end{aligned}$$

with

$$\langle I\mu^2 \rangle \sim P = \langle \mu^2 \mathcal{B} \rangle = P_+ + P_-$$

and

$$\begin{aligned} \chi_+ &= \chi_+(f_+), \quad f_+ = \frac{F_+}{E_+} \in [0, 1], \\ \chi_- &= \chi_-(f_-), \quad f_- = \frac{F_-}{E_-} \in [-1, 0], \\ \chi_+ &= \frac{8 f_+^2}{1 + \sqrt{1 + 12 f_+ - 12 f_+^2} + 6 f_+} \in [0, 1], \\ \chi_- &= \frac{8 f_-^2}{1 + \sqrt{1 - 12 f_- - 12 f_-^2} - 6 f_-} \in [0, 1]. \end{aligned}$$

We consider the transport equation as before and integrate to obtain the balance equations

$$\left\langle \left[I_t + \mu I_x - \frac{1}{\varepsilon}(B(I) - I) \right] m \right\rangle = 0,$$

with

$$m = (1^-, \mu^-, 1^+, \mu^+).$$

Note that $\langle B(I) \rangle = E_+ + E_- = \langle I \rangle$. The system reads finally

$$\begin{aligned}
\epsilon \partial_t E_- + \partial_x F_- &= \frac{1}{\varepsilon}\left[\frac{E_+ + E_-}{2} - E_- \right], \\
\epsilon \partial_t F_- + \partial_x \chi_- E_- &= \frac{1}{\varepsilon}\left[-\frac{E_+ + E_-}{4} - F_- \right], \\
\epsilon \partial_t E_+ + \partial_x F_+ &= \frac{1}{\varepsilon}\left[\frac{E_+ + E_-}{2} - E_+ \right], \\
\epsilon \partial_t F_+ + \partial_x \chi_+ E_+ &= \frac{1}{\varepsilon}\left[\frac{E_+ + E_-}{4} - F_+ \right].
\end{aligned} \tag{9}$$

The eigenvalues associated with the first two variables are always negative, and the other two always positive. We use again the Maxwell procedure to find boundary conditions for our moment system,

$$\int_0^\infty \int_{\mu>0} \mu I_b m(\mu)\, d\mu\, d\nu = \int_0^\infty \int_{\mu>0} \mu \mathcal{B} m(\mu)\, d\mu\, d\nu,$$

with $m = \binom{1}{\mu}$. This gives at $x = 0$, where two boundary conditions are needed according to two positive eigenvalues of the system,

$$\begin{aligned}
F^+ &= \langle \mu^+ I_b \rangle, \\
P^+ &= \langle (\mu^+)^2 I_b \rangle,
\end{aligned} \tag{10}$$

and at $x = 1$

$$\begin{aligned}
F^- &= \langle \mu_- I_b \rangle, \\
P^- &= \langle (\mu^-)^2 I_b \rangle.
\end{aligned} \tag{11}$$

In case of the above example with equilibrium boundary conditions we have at $x = 0$

$$F^+ = 0, \quad P^+ = 0,$$

and at $x = 1$

$$F^- = -\frac{1}{4}, \quad P^- = \frac{1}{6}.$$

Obviously the number of boundary conditions is fixed and does not depend on the special physical situation, as in the case of the full moment method.

An asymptotic expansion $E_+ = E_+^0 + \epsilon E_+^1 + \cdots$, etc., for the continuous model

$$\varepsilon \partial_t E_\mp + \partial_x F_\mp = \frac{1}{\varepsilon}\left[\frac{E_+ + E_-}{2} - E_\mp\right],$$

$$\varepsilon \partial_t F_\mp + \partial_x P_\mp = \frac{1}{\varepsilon}\left[\mp\frac{E_+ + E_-}{4} - F_\mp\right]$$

gives to order 0

$$\begin{aligned} E_\mp^0 &= \frac{E_+^0 + E_-^0}{2} = \frac{E^0}{2}, \\ F_\mp^0 &= \mp\frac{E_+^0 + E_-^0}{4} = \mp\frac{E^0}{4}, \\ f_\mp^0 &= \mp\frac{1}{2}, \end{aligned}$$

and

$$P_\mp^0 = \chi_\mp(f_\mp^0)E_\mp^0 = \frac{1}{3}E_\mp^0$$

or

$$P^0 = \frac{1}{3}E^0.$$

To order 1 it yields

$$\partial_x P_\mp^0 = \mp\frac{E_+^1 + E_-^1}{4} - F_\mp^1.$$

By summing these two equations we get

$$\partial_x P^0 = -F^1.$$

Order 2 yields

$$\partial_t E_\mp^0 + \partial_x F_\mp^1 = \frac{1}{\varepsilon}\left[\frac{E_+^2 + E_-^2}{2} - E_\mp^2\right].$$

By summing,

$$\partial_t E^0 + \partial_x F^1 = 0$$

or

$$\partial_t E^0 - \partial_x\left(\frac{1}{3}\partial_x E^0\right) = 0.$$

That means we have obtained again the classical diffusion equation.

3. A NUMERICAL METHOD AND NUMERICAL RESULTS

We discretize the equations using a simple and straightforward upwind scheme. The discrete quantities at $t_n = n\Delta t$ and $x_i = i\Delta x$ we denote E_i^n, etc. Due to the sign of the eigenvalues one obtains

$$E_{-,i}^{n+1} - E_{-,i}^{n} + \frac{\Delta t}{\epsilon\Delta x}\left(F_{-,i+1}^{n} - F_{-,i}^{n}\right) = \frac{\Delta t}{\varepsilon^2}\left[\frac{E_{+,i}^{n} + E_{-,i}^{n}}{2} - E_{-,i}^{n}\right],$$

$$F_{-,i}^{n+1} - F_{-,i}^{n} + \frac{\Delta t}{\epsilon\Delta x}\left(\chi_- E_{-,i+1}^{n} - \chi_- E_{-,i}^{n}\right) = \frac{\Delta t}{\varepsilon^2}\left[-\frac{E_{+,i}^{n} + E_{-,i}^{n}}{4} - F_{-,i}^{n}\right],$$

$$E_{+,i}^{n+1} - E_{+,i}^{n} + \frac{\Delta t}{\epsilon\Delta x}\left(F_{+,i}^{n} - F_{+,i-1}^{n}\right) = \frac{\Delta t}{\varepsilon^2}\left[\frac{E_{+,i}^{n} + E_{-,i}^{n}}{2} - E_{+,i}^{n}\right],$$

$$F_{+,i}^{n+1} - F_{+,i}^{n} + \frac{\Delta t}{\epsilon\Delta x}\left(\chi_+ E_{+,i}^{n} - \chi_+ E_{+,i-1}^{n}\right) = \frac{\Delta t}{\varepsilon^2}\left[\frac{E_{+,i}^{n} + E_{-,i}^{n}}{4} - F_{+,i}^{n}\right].$$

As boundary conditions, F_+ and $P_+ = \chi_+ E_+$ are described on the left hand side of the boundary and F_- and P_+ on the right hand side. An asymptotic expansion $E_{-,i}^{n} = E_{-,i}^{0,n} + \varepsilon E_{-,i}^{1,n} + \varepsilon^2 E_{-,i}^{2,n} + \cdots$, etc., gives to order 0

$$E_{\mp,i}^{0,n} = \frac{E_{+,i}^{0,n} + E_{-,i}^{0,n}}{2} = \frac{E_i^{0,n}}{2},$$

$$F_{\mp,i}^{0,n} = \mp\frac{E_{+,i}^{0,n} + E_{-,i}^{0,n}}{4} = \mp\frac{E_i^{0,n}}{4}.$$

Moreover,

$$f_{\mp,i}^{0,n} = \mp\frac{1}{2}$$

and

$$P_{\mp,i}^{0,n} = \frac{1}{3}\frac{E_i^{0,n}}{2}.$$

To order 1 the equations are

$$\mp\frac{1}{\Delta x}\left(P_{\mp,i}^{0,n} - P_{\mp,i\pm1}^{0,n}\right) = \mp\frac{E_{+,i}^{1,n} + E_{-,i}^{1,n}}{4} - F_{\mp,i}^{1,n}.$$

Since

$$E_{+,i}^{1,n} + E_{-,i}^{1,n} = 0,$$

we obtain

$$\mp\frac{1}{\Delta x}\left(P_{\mp,i}^{0,n} - P_{\mp,i\pm1}^{0,n}\right) = \mp\frac{1}{6\Delta x}\left(E_i^{0,n} - E_{i\pm1}^{0,n}\right) = -F_{\mp,i}^{1,n}.$$

Order 2 yields

$$\frac{1}{\Delta t}\left(E_i^{0,n+1}-E_i^{0,n}\right)+\frac{1}{\Delta x}\left(F_{-,i+1}^{1,n}-F_{-,i}^{1,n}\right)+\frac{1}{\Delta x}\left(F_{+,i}^{1,n}-F_{+,i-1}^{1,n}\right)=0$$

or

$$\frac{1}{\Delta t}\left(E_i^{0,n+1}-E_i^{0,n}\right)-\frac{1}{6\Delta x^2}\left(\left(E_{i+2}^{0,n}-E_{i+1}^{0,n}\right)-\left(E_{i+1}^{0,n}-E_i^{0,n}\right)\right)$$
$$-\frac{1}{6\Delta x^2}\left(\left(E_i^{0,n}-E_{i+1}^{0,n}\right)-\left(E_{i-1}^{0,n}-E_{i-2}^{0,n}\right)\right)=0,$$

which is equivalent to

$$\frac{1}{\Delta t}\left(E_i^{0,n+1}-E_i^{0,n}\right)-\frac{1}{3}\left[\frac{1}{2}\left(\frac{1}{\Delta x^2}\left(E_{i+2}^{0,n}-2E_{i+1}^{0,n}+E_i^{0,n}\right)\right.\right.$$
$$\left.\left.+\frac{1}{\Delta x^2}\left(E_i^{0,n}-2E_{i-1}^{0,n}+E_{i-2}^{0,n}\right)\right)\right]=0.$$

This is up to $O(\Delta x^2)$ an approximation of the time discretized diffusion equation.

In the first four figures we show the solutions of the different models for a stationary situation using the example described in the text with equilibrium boundary conditions (6) and $\delta=\delta_r$ at $x=1$ and $\delta=\delta_l$ at $x=0$, respectively. In the first four plots we show only the sensitive boundary region $x\in[0,0.1]$ and not the full domain of computation.

Figure 1 shows the comparison of the densities for the case $\delta_l=0$, $\delta_r=0$, and $\epsilon=0.1$. Figure 2 shows the comparison of the relative fluxes f. The half space moment expansion (9) with boundary conditions (10, 11) is compared with the diffusion solution with slip boundary conditions derived from the Maxwell procedure with the distribution function (8) and the full transport solution. Moreover, the solution of the full moment model is plotted. According to the discussion in the introduction, the boundary conditions are inconsistent with these equations. To circumvent this problem we use a kinetic scheme that includes the kinetic boundary conditions directly in the numerical method. See [1, 7, 13] for similiar approaches and the discussion above.

Figures 3 and 4 show the comparison of the relative fluxes for $\delta_l=0.02$ and again $\delta_r=1.0$. We include for comparison the solution of the full space moment method. In these cases the first Marshak condition can be used as boundary condition for the full moment equation. Since the relative flux is in this case in the subcritical domain, this fits to the required number of boundary conditions for the full moment system. The results for the full moment model obtained with a standard method are in these cases the same as those obtained with the above-mentioned kinetic scheme.

Figure 5 shows the results of an extension of the above equations to the case with absorption. We consider an example with two beams. Equation (1) is considered in the form

$$\epsilon\partial_t I+\mu\partial_x I=\frac{1}{\varepsilon}(B(I)-I)-\sigma_a\epsilon I. \tag{12}$$

The boundary conditions are $\delta_l=\delta_r=1$, $\epsilon=0.02$, and $\sigma_a\epsilon^2=1$. The corresponding approximate equations are easily deduced. For further comparison of the results we refer to [1].

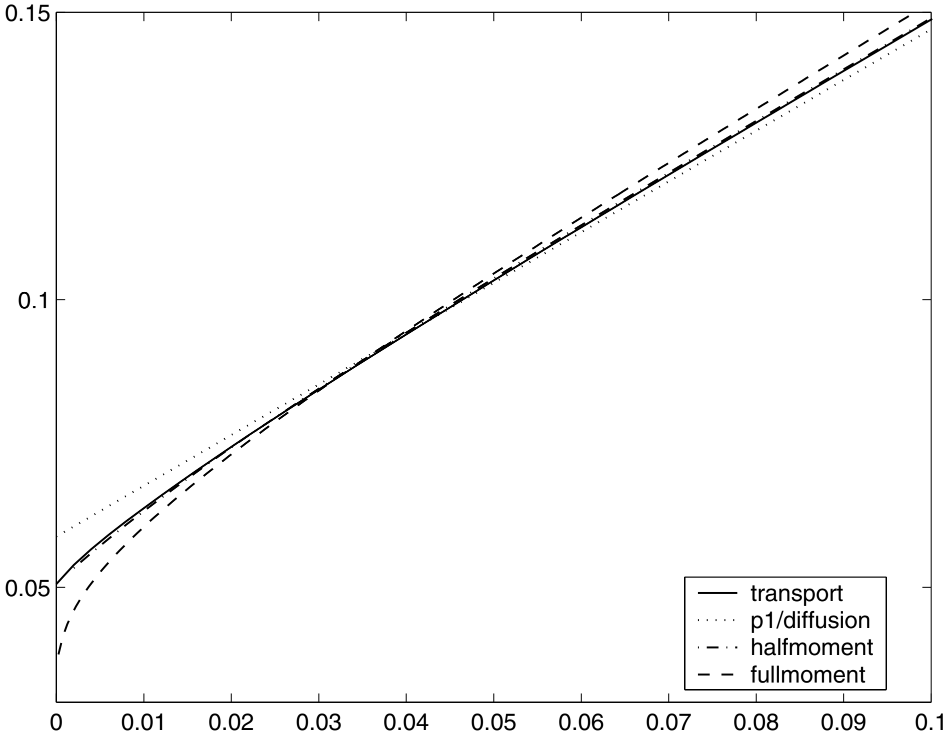

FIG. 1. Density E, $\epsilon = 0.1$, $\delta_l = 0$, $\delta_r = 1$.

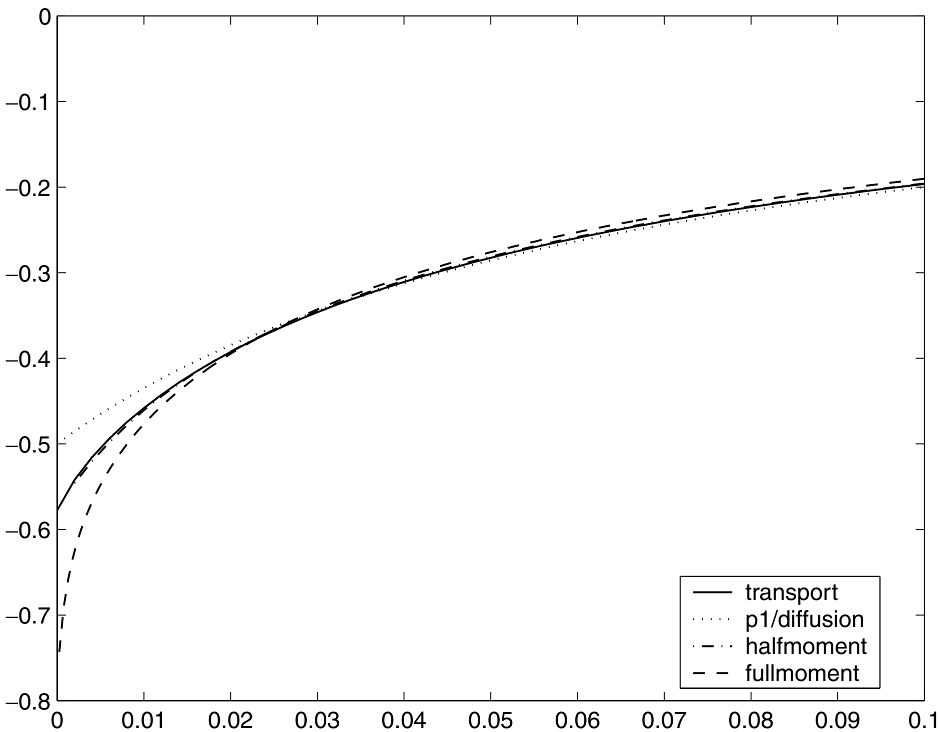

FIG. 2. Relative flux f, $\epsilon = 0.1$, $\delta_l = 0$, $\delta_r = 1$.

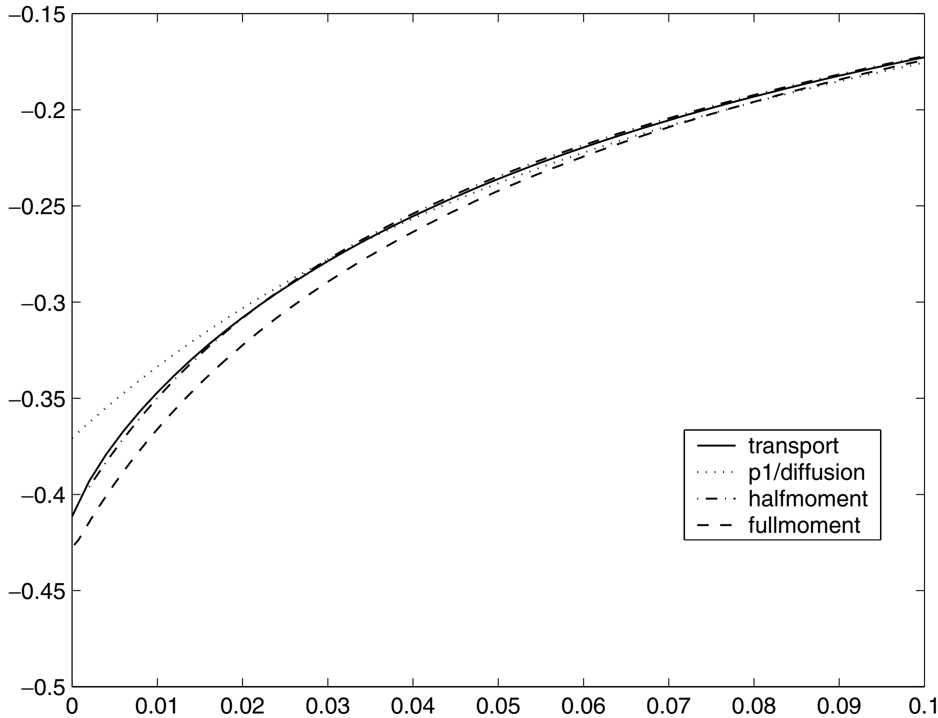

FIG. 3. Relative flux f, $\epsilon = 0.1$, $\delta_l = 0.02$, $\delta_r = 1$.

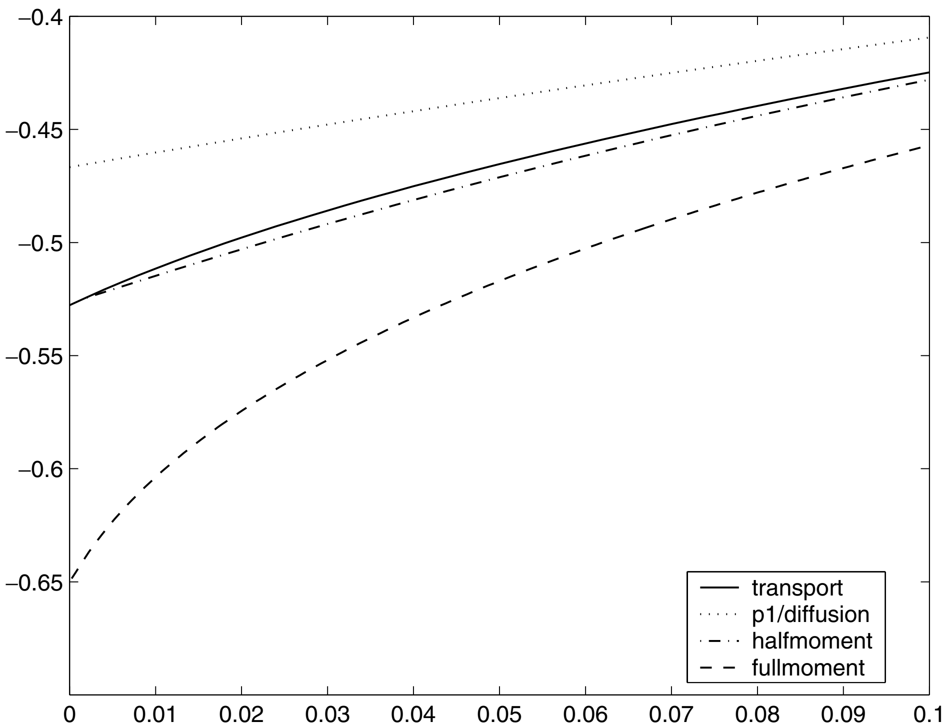

FIG. 4. Relative flux f, $\epsilon = 1$, $\delta_l = 0.02$, $\delta_r = 1$.

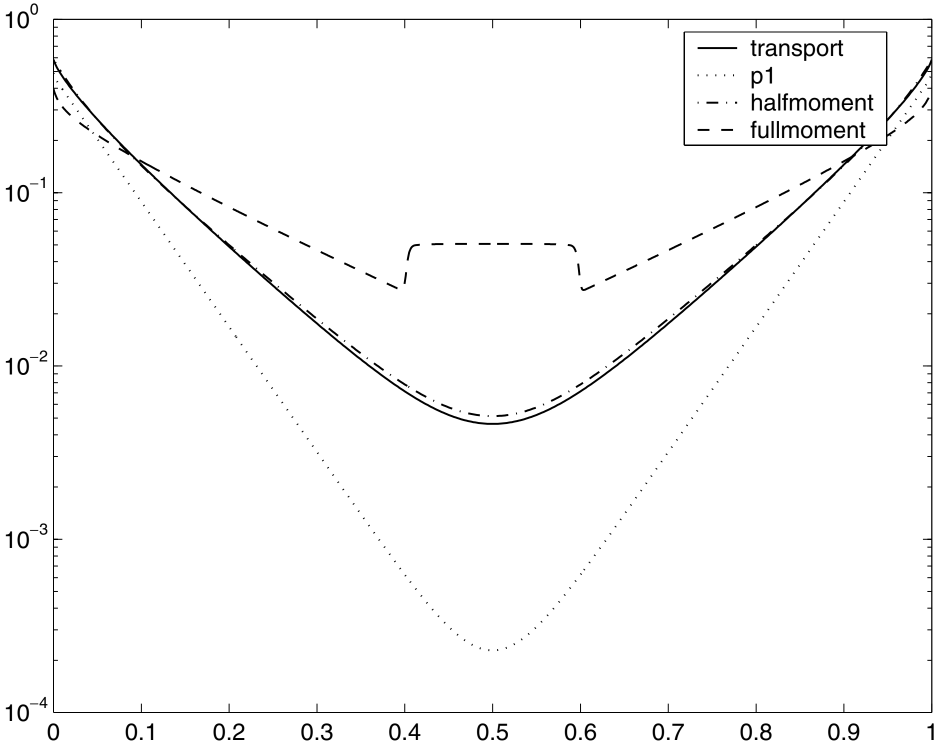

FIG. 5. Density E, $\epsilon = 0.02$, $\delta_l = \delta_r = 1$, and $\sigma_a \epsilon^2 = 1$.

One observes that the density and the flux are in all cases considered here much better approximated by the half space moment method than by the diffusion equation with slip boundary coefficients or by the full space moment method. This is true not only for boundary regions, but also for problems like the two-beam problem discussed in the last experiment. In this case interior numerical shocks created by the full moment method are observed (see [1]).

4. SUMMARY

- A half space moment method based on entropy minimization is developed for a one-dimensional simplified model.
- The results show that the solution fits in the cases considered here nearly perfectly well with the solution of the full radiative transfer equation.
- Connections and differences to kinetic schemes including kinetic boundary conditions in the numerical solution of the full space moment model are discussed.
- The extension to cases with absorption is straightforward and has been treated in the paper.

5. EXTENSIONS

- The above model is easily used in combination with a heat transfer equation and general radiative mean absorption coefficients [5]. For example, one may consider the equations

$$\epsilon^2 \partial_t T = \epsilon^2 k \partial_x^2 T + \langle I \rangle - \langle B(T) \rangle,$$
$$\epsilon^2 \partial_t I + \epsilon \mu \partial_x I = B(T) - I,$$

with

$$B(T) = \frac{15}{\pi^4} \frac{\nu^3}{\exp(c\nu) - 1} \quad \text{with } c = T^{-1/4}.$$

- More-complicated wall models like reflection boundary conditions can be included into the model numerically.
- Frequency dependence of the coefficients, for example piecewise constant absorption coefficients, can be taken into account by considering the relevant frequency bands separately. However, the model has to be closed numerically in this case. For an example for a full moment closure including frequency bands, we refer the reader to [14].
- Extensions to multidimensional cases are based on directional splitting. The extension to several dimensions will be investigated in a following paper.

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